APPROXIMATION AND LIMIT THEOREMS FOR QUANTUM STOCHASTIC MODELS WITH UNBOUNDED COEFFICIENTS

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ABSTRACT. We prove a limit theorem for quantum stochastic differential equations with unbounded coefficients which extends the Trotter-Kato theorem for contraction semigroups. From this theorem, general results on the convergence of approximations and singular perturbations are obtained. The results are illustrated in several examples of physical interest.

1. INTRODUCTION

It has been well established that the quantum stochastic differential equations of Hudson and Parthasarathy [18] provide an extremely rich source of realistic physical models, particularly in the field of quantum optics [2]. The physical models in this framework are always Markovian, just like their classical counterparts, and it is the Markov property which makes these models particularly tractable. Several authors have developed mathematical methodology which can be used to obtain these Markovian models directly from fundamental models in quantum field theory by applying a suitable limiting procedure [1, 14, 6], which places such models on a firm physical and mathematical foundation. On the other hand, the simplification provided by the Markov property is essential in many studies, particularly in investigations where nontrivial dynamics plays an important role.

Despite the significant simplification provided by the Markov limit, many realistic physical models remain rather complex, and often further simplifications are sought. In the physics literature it has been known for a long time that models with multiple time scales can be significantly simplified, in parameter regimes where these time scales are well separated, by eliminating the fast variables from the description. Such singular perturbation methods are known as 'adiabatic elimination' in the physics literature, and are extremely widespread on a heuristic level. Until recently, however, no riguruous development was available in the quantum probability literature. In two recent papers [15, 3] initial steps were made in this direction, but, as is discussed below, these results are not entirely satisfactory.

The motivation for the current paper stems from the goal to develop a widely applicable singular perturbation theory for quantum stochastic differential equations. As in previous results on the Markov limit, the simplified model is obtained by taking the limit of a sequence of quantum stochastic differential equations which depend on a parameter. Our main result, theorem 2 below, provides a general

method to prove the convergence of a sequence of quantum stochastic differential equations to a limiting equation. This technique hinges crucially on the Markov property of all the equations involved, and extends the Trotter-Kato theorem in the theory of one-parameter semigroups [5]. Using this technique, we develop a general singular perturbation method for quantum stochastic differential equations, theorem 3 below, which contains and extends all known results on this topic.

Relation to previous work. To our knowledge, the first rigorous result on singular perturbation for quantum stochastic differential equations was given in the paper [15]. This paper considers an atomic system inside an optical cavity, where the cavity is strongly coupled to an external field. It is shown that in the strong coupling limit, the cavity can be entirely eliminated from the model leaving an effective interaction between the atoms and the external field. The convergence to the limiting equation is given in the weak topology, and the proofs utilise Dyson expansion techniques similar to those used in the literature on Markov limits. However, the results in this paper are limited to the particular model under consideration, and it is unclear how to generalize these techniques to solve general singular perturbation problems.

In [3] singular perturbations are studied in a general setting. The key observation in this paper is that, due to the Markovian nature of all the equations, the Trotter-Kato theorem can be used to prove strong convergence. This is done by defining certain skew semigroups on the algebra of bounded operators on the initial space. Though results of a general nature are obtained in this fashion, the technique is essentially restricted to the case that all quantum stochastic differential equations of interest have bounded coefficients. In the unbounded case the semigroups on the algebra of bounded operators are typically only weak* continuous and not strongly continuous, which precludes the application of the Trotter-Kato theorem. Unfortunately, the boundedness assumption excludes a large number of singular perturbation problems of practical interest, including the problem studied in [15].

The singular perturbation results in this paper extend and bridge the gap between the results of [15, 3]. We consider a family of quantum stochastic differential equations whose coefficients are not necessarily bounded but possess a common invariant domain. Rather than considering semigroups on the algebra of bounded operators, we work directly with semigroups on the initial Hilbert space. This approach has many far reaching consequences. Unbounded coefficients are easily treated in this setting, and the type of convergence which we obtain (strong convergence uniformly on compact time intervals) is much stronger than in [15, 3]. Moreover, the proofs in this setting are significantly simplified as compared to [15, 3], which highlights that the current approach is very natural. In order to prove our singular perturbation results, we first prove a general theorem on the convergence of quantum stochastic differential equations with unbounded coefficients, which extends the Trotter-Kato theorem to our setting. With this theorem in hand, the remaining work on singular perturbations is chiefly algebraic (rather than analytic) in nature.

Though motivated by a rather different set of problems, our main convergence result, theorem 2, is closely related to results obtained for the purpose of proving existence and uniqueness of solutions of quantum stochastic differential equations with unbounded coefficients [11, proposition 3.4] and [21, theorem 1.7], [20]. Instead of showing the existence of a contractive cocycle satisfying the limit equation, we

assume the existence of a unitary cocycle satisfying the limit equation, which allows us to obtain a much stronger form of convergence than is obtained in [11, 21, 20].

Organization of the paper. The remainder of this paper is organized as follows. In section 2 we introduce the class of models that we will consider and the technical conditions which are assumed to be in force throughout the paper. In subsection 2.1 we describe our main convergence result, and subsection 2.2 describes our general result on singular perturbations. Section 3 is devoted to the discussion of several examples of physical interest which demonstrate the flexibility of our results. Finally, section 4 presents the proofs of the results described in subsection 2.1, while section 5 presents the proofs of the results described in section 2.2.

2. Main results

Throughout this paper we let $\mathcal{H}$, the initial space, be a separable (complex) Hilbert space. We denote by $\mathcal{F} = \Gamma_s(L^2(\mathbb{R}_+; \mathbb{C}^n))$ the symmetric Fock space with multiplicity $n \in \mathbb{N}$ (i.e., the one-particle space is $\mathbb{C}^n \otimes L^2(\mathbb{R}_+) \cong L^2(\mathbb{R}_+; \mathbb{C}^n)$), and by $e(f)$, $f \in L^2(\mathbb{R}_+; \mathbb{C}^n)$ the exponential vectors in $\mathcal{F}$. The annhiliation, creation and gauge processes on $\mathcal{F}$, as well as their ampliations to $\mathcal{H} \otimes \mathcal{F}$, will be denoted as A_t^i, $A_t^{i\dagger}$ and Λ_t^{ij}, respectively (the channel indices are relative to the canonical basis of $\mathbb{C}^n$). Moreover, we will fix once and for all a dense domain $\mathcal{D} \subset \mathcal{H}$ and a dense domain of exponential vectors $\mathcal{E} = \mathrm{span}\{e(f) : f \in \mathfrak{S}\} \subset \mathcal{F}$, where $\mathfrak{S} \subset L^2(\mathbb{R}_+; \mathbb{C}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{C}^n)$ is an admissible subspace in the sense of Hudson-Parthasarathy [18] which is presumed to contain at least all simple functions. For a detailed description of these definitions and of the Hudson-Parthasarathy stochastic calculus which we will use throughout this paper, we refer to [18, 22, 2].

For every $k \in \mathbb{N}$ we consider a quantum stochastic differential equation of the form

(1)
$$dU_t^{(k)} = U_t^{(k)} \left\{ \sum_{i,j=1}^n \left(N_{ij}^{(k)} - \delta_{ij}\right) d\Lambda_t^{ij} + \sum_{i=1}^n M_i^{(k)} dA_t^{i\dagger} + \sum_{i=1}^n L_i^{(k)} dA_t^i + K^{(k)} dt \right\},$$

where $U_0^{(k)} = I$ and the quantum stochastic integrals are defined relative to the domain $\mathcal{D} \overline{\otimes} \mathcal{E}$. The purpose of this paper is to prove that, when the dependence of the coefficients on k is chosen appropriately, the solutions $U_t^{(k)}$ converge as $k \to \infty$ in a suitable sense to the solution of a limit quantum stochastic differential equation

(2)
$$dU_t = U_t \left\{ \sum_{i,j=1}^n \left(N_{ij} - \delta_{ij}\right) d\Lambda_t^{ij} + \sum_{i=1}^n M_i \, dA_t^{i\dagger} + \sum_{i=1}^n L_i \, dA_t^i + K \, dt \right\}.$$

Of course, the form of K, etc. will depend on our choice for $K^{(k)}$, etc., and a part of our task will be to identify the appropriate limit coefficients. We will both prove a general convergence result—a version of the Trotter-Kato theorem for quantum stochastic differential equations—and investigate in detail a large class of singular perturbation problems (known as *adiabatic elimination* in the physics literature) which are of significant practical importance in obtaining tractable simplifications of complex physical models.

As it turns out, it is not always the case that $U_t^{(k)}$ converges on the entire Hilbert space $\mathcal{H} \otimes \mathcal{F}$; in singular perturbation problems the limit will only exist on a closed

subspace $\mathcal{H}_0 \otimes \mathcal{F}$, while for vectors in $\mathcal{H}_0^\perp \otimes \mathcal{F}$ the limit becomes increasingly singular as $k \to \infty$. We will return to this phenomenon in detail later on; for the time being, suffice it to say that we must take into account the possibility that the limiting equation (2) lives on a smaller space $\mathcal{H}_0 \otimes \mathcal{F}$ than the prelimit equations (1). For the time being, we will fix the closed subspace $\mathcal{H}_0 \subset \mathcal{H}$ and the dense domain $\mathcal{D}_0 \subset \mathcal{D} \cap \mathcal{H}_0$ in $\mathcal{H}_0$. The quantum stochastic integrals in (2) will then be defined relative to the domain $\mathcal{D}_0 \,\overline{\otimes}\, \mathcal{E}$.

We begin by imposing conditions on these equations that are assumed to be in force throughout this paper. For notational simplicity, we use the same notation for operators on $\mathcal{H}$ or on $\mathcal{F}$ and for their ampliations to $\mathcal{H} \otimes \mathcal{F}$ (and similarly for $\mathcal{H}_0$ and $\mathcal{H}_0 \otimes \mathcal{F}$, etc.) Throughout, we denote by † an adjoint relationship on the domain of definition, i.e., two operators $X, X^\dagger$ on the domain $\mathcal{D}$ (or $\mathcal{D}_0$) are always presumed to satisfy the adjoint pair property $\langle u, Xv \rangle = \langle X^\dagger u, v \rangle$ for all $u, v \in \mathcal{D}$ (resp. $\mathcal{D}_0$).

Condition 1 (Hudson-Parthasarathy conditions)**.** *For all $k \in \mathbb{N}$ and $1 \le i, j \le n$, the operators $K^{(k)}$, $K^{(k)\dagger}$, $L_i^{(k)}$, $L_i^{(k)\dagger}$, $M_i^{(k)}$, $M_i^{(k)\dagger}$, $N_{ij}^{(k)}$, $N_{ij}^{(k)\dagger}$ have the common invariant domain $\mathcal{D}$, and the operators K, $K^\dagger$, L_i, $L_i^\dagger$, M_i, $M_i^\dagger$, N_{ij}, $N_{ij}^\dagger$ have the common invariant domain $\mathcal{D}_0$. In addition, we assume that the Hudson-Parthasarathy conditions*

$$K + K^\dagger = -\sum_{i=1}^{n} L_i L_i^\dagger, \qquad M_i = -\sum_{j=1}^{n} N_{ij} L_j^\dagger,$$

$$\sum_{j=1}^{n} N_{mj} N_{\ell j}^\dagger = \sum_{j=1}^{n} N_{jm}^\dagger N_{j\ell} = \delta_{m\ell}$$

are satisfied on $\mathcal{D}_0$, and that

$$K^{(k)} + K^{(k)\dagger} = -\sum_{i=1}^{n} L_i^{(k)} L_i^{(k)\dagger}, \qquad M_i^{(k)} = -\sum_{j=1}^{n} N_{ij}^{(k)} L_j^{(k)\dagger},$$

$$\sum_{j=1}^{n} N_{mj}^{(k)} N_{\ell j}^{(k)\dagger} = \sum_{j=1}^{n} N_{jm}^{(k)\dagger} N_{j\ell}^{(k)} = \delta_{m\ell}$$

are satisfied on $\mathcal{D}$ for all $k \in \mathbb{N}$.

Denote by $\theta_t : L^2([t, \infty[; \mathbb{C}^n) \to L^2(\mathbb{R}_+; \mathbb{C}^n)$ the canonical shift $\theta_t f(s) = f(t+s)$, and by $\Theta_t : \mathcal{F}_{[t} \to \mathcal{F}$ its second quantization (here $\mathcal{F} \cong \mathcal{F}_{t]} \otimes \mathcal{F}_{[t}$ denotes the usual continuous tensor product decomposition). Recall that an adapted process $\{U_t : t \ge 0\}$ on $\mathcal{H} \otimes \mathcal{F}$ is called a *contraction cocycle* if U_t is a contraction for all $t \ge 0$, $t \mapsto U_t$ is strongly continuous and $U_{s+t} = U_s(I \otimes \Theta_s^* U_t \Theta_s)$ with $I \otimes \Theta_s^* U_t \Theta_s \in \mathcal{F}_{s]} \otimes \mathcal{H} \otimes \mathcal{F}_{[s} \cong \mathcal{H} \otimes \mathcal{F}$. If U_t, $t \ge 0$ are even unitary, the process is called a *unitary cocycle*.

Condition 2 (Cocycle solutions)**.** *For all $k \in \mathbb{N}$, equation (1) possesses a unique solution $\{U_t^{(k)} : t \ge 0\}$ which extends to a contraction cocycle on $\mathcal{H} \otimes \mathcal{F}$, while equation (2) possesses a unique solution $\{U_t : t \ge 0\}$ which extends to a unitary cocycle on $\mathcal{H}_0 \otimes \mathcal{F}$.*

The following elementary result is proved in section 4.

Lemma 1. *For $\alpha, \beta \in \mathbb{C}^n$, define $T_t^{(\alpha\beta)} : \mathcal{H}_0 \to \mathcal{H}_0$ such that*

$$\langle u, T_t^{(\alpha\beta)} v \rangle = e^{-(|\alpha|^2 + |\beta|^2)t/2} \langle u \otimes e(\alpha I_{[0,t]}), U_t\, v \otimes e(\beta I_{[0,t]}) \rangle \qquad \forall u, v \in \mathcal{H}_0,\ t \geq 0.$$

Then $T_t^{(\alpha\beta)}$ is a strongly continuous contraction semigroup on $\mathcal{H}_0$, and the generator $\mathscr{L}^{(\alpha\beta)}$ of this semigroup satisfies $\mathrm{Dom}(\mathscr{L}^{(\alpha\beta)}) \supset \mathcal{D}_0$ such that for $u \in \mathcal{D}_0$

$$(3) \qquad \mathscr{L}^{(\alpha\beta)} u = \left(\sum_{i,j=1}^{n} \alpha_i^* N_{ij} \beta_j + \sum_{i=1}^{n} \alpha_i^* M_i + \sum_{i=1}^{n} L_i \beta_i + K - \frac{|\alpha|^2 + |\beta|^2}{2} \right) u.$$

The same result holds for $T_t^{(k;\alpha\beta)} : \mathcal{H} \to \mathcal{H}$ and $\mathscr{L}^{(k;\alpha\beta)}$, defined by replacing U_t by $U_t^{(k)}$ and making the obvious modifications. In particular, $\mathrm{Dom}(\mathscr{L}^{(k;\alpha\beta)}) \supset \mathcal{D}$.

We now impose one further condition: we assume that (3) completely determines $\mathscr{L}^{(\alpha\beta)}$.

Condition 3 (Core)**.** *For any $\alpha, \beta \in \mathbb{C}^n$, the domain $\mathcal{D}_0$ is a core for $\mathscr{L}^{(\alpha\beta)}$.*

Remark 1. Note that it is not necessary for our purposes to assume that $U_t^{(k)}$ is unitary or that $\mathcal{D}$ is a core for $\mathscr{L}^{(k;\alpha\beta)}$, $k \in \mathbb{N}$. Typically this will nonetheless be the case in physical applications; in particular, recall that when the coefficients are bounded, condition 1 guarantees that the solution of the associated quantum stochastic differential equation is unitary and the core condition is trivially satisfied. In the unbounded case, however, the solution may only be a contraction due to the possibility of explosion.

Remark 2. We have chosen the *right* Hudson-Parthasarathy equations (1) and (2), rather than the more familiar *left* equations where the coefficients are placed to the left of the solution. This means that the Schrödinger evolution of a state vector $\psi \in \mathcal{H}_0 \otimes \mathcal{F}$ is given by $U_t^* \psi$, etc. The main reason for this choice is that for quantum stochastic differential equations with unbounded coefficients, it is generally much easier to prove the existence of a unique cocycle solution for the right equation than for the left equation (see [10, 20]); this is chiefly because it is not clear that the solution should leave $\mathcal{D}_0 \overline{\otimes} \mathcal{E}$ invariant, so that the left equation may not be well defined (this is not a problem for the right equation, as the solution only appears to the left of all unbounded coefficients and is presumed to be bounded). This appears to be an artefact of the fact that quantum stochastic integrals are defined on a fixed domain, and is not necessarily a physical problem.

We will ultimately prove that $U_t^{(k)*}$ converges to U_t^* strongly on $\mathcal{H}_0 \otimes \mathcal{F}$ uniformly on compact time intervals, i.e., we will show that

$$\lim_{k \to \infty} \sup_{0 \leq t \leq T} \|U_t^{(k)*} \psi - U_t^* \psi\| = 0 \qquad \forall \psi \in \mathcal{H}_0 \otimes \mathcal{F},\ T < \infty.$$

Therefore, there is no loss of generality in working with the more tractable right equations. If we wish to begin with a well defined left equation, our results can be immediately applied to the Hudson-Parthasarathy equation for its adjoint.

Remark 3. In practice, the result of Fagnola [10, 12] provides a convenient sufficient condition which can be used to verify conditions 2 and 3. Let us recall a slightly stronger version of this result (for simplicity phrased in terms of U_t, the analogous result holds for $U_t^{(k)}$). Suppose that for all $u \in \mathcal{D}_0$ and $\ell \in \mathbb{N}$, there exists a constant $c(\ell, u)$ such that:

(a) For all $u \in \mathcal{D}_0$ and for some $\varepsilon > 0$ independent of u

$$\sum_{\ell=1}^{\infty} c(\ell, u)\, \varepsilon^{\ell} < \infty;$$

(b) For all $\ell \in \mathbb{N}$ and all choices of $X(1), \ldots, X(\ell)$, where each $X(\cdot)$ is one of K, $K^{\dagger}$, L_i, $L_i^{\dagger}$, M_i, $M_i^{\dagger}$, N_{ij}, $N_{ij}^{\dagger}$, we have for all $u \in \mathcal{D}_0$

$$\|X(1)\cdots X(\ell)u\| \leq c(\ell, u)\, \sqrt{(\ell + m)!},$$

where m is the number of occurences of K or $K^{\dagger}$ in the sequence $X(1), \ldots, X(\ell)$.

Then equation (2) possesses a unique solution $\{U_t : t \geq 0\}$ that extends to a unitary cocycle on $\mathcal{H}_0 \otimes \mathcal{F}$, which verifies condition 2. To show that $\mathcal{D}_0$ is a core for $\mathscr{L}^{(\alpha\beta)}$, it suffices to note that the Fagnola conditions imply that every $u \in \mathcal{D}_0$ is analytic [4, definition 3.1.17], so that the result follows from [4, corollary 3.1.20]. This verifies condition 3.

If we are only interested in the existence and uniqueness of a contraction cocycle U_t, simple sufficient conditions exist in a much more general setting than the one considered in [10, 12]. In particular, from [20, theorem 1.1], one can see that it suffices to prove that the closure of $\mathscr{L}^{(\alpha\beta)}$ on $\mathcal{D}_0$, as defined in equation (3), is the generator of a strongly continuous contraction semigroup on $\mathcal{H}_0$ for every $\alpha, \beta \in \mathbb{C}^n$.

2.1. A Trotter-Kato theorem for quantum stochastic differential equations.

In order to prove that $U_t^{(k)}$ converges to U_t as $k \to \infty$, we would like to use an argument which requires only to verify conditions on the coefficients of the associated equations. After all, the dependence of the coefficients on k is known in advance, while the solutions of the equations could potentially depend on k in a complicated manner. In order to set the stage for our main convergence result, let us shift our attention for the moment to the semigroups $T_t^{(k;\alpha\beta)}$ and $T_t^{(\alpha\beta)}$. The convergence of the former to the latter can be verified using a standard technique: the Trotter-Kato theorem. For future reference, we state this result here in a particularly convenient form [5, theorem 3.17, p. 80].

Theorem 1 (Trotter-Kato). *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}_0 \subset \mathcal{H}$ be a closed subspace. For each $k \in \mathbb{N}$, let $T_t^{(k)}$ be a strongly continuous contraction semigroup on $\mathcal{H}$ with generator $\mathscr{L}^{(k)}$. Moreover, let T_t be a strongly continuous contraction semigroup on $\mathcal{H}_0$ with generator $\mathscr{L}$. Let $\mathcal{D}_0$ be a core for $\mathscr{L}$. The following conditions are equivalent:*

(a) *For all $\psi \in \mathcal{D}_0$ there exist $\psi^{(k)} \in \mathrm{Dom}(\mathscr{L}^{(k)})$ such that*

$$\psi^{(k)} \xrightarrow{k \to \infty} \psi, \qquad \mathscr{L}^{(k)}\psi^{(k)} \xrightarrow{k \to \infty} \mathscr{L}\psi.$$

(b) *For all $T < \infty$ and $\psi \in \mathcal{H}_0$*

$$\lim_{k \to \infty} \sup_{0 \leq t \leq T} \|T_t^{(k)}\psi - T_t\psi\| = 0.$$

It should be noted, in particular, that the condition (a) depends directly on the generators $\mathscr{L}^{(k)}$ and $\mathscr{L}$, whose form (in terms of k) is typically known explicitly. Condition (b), however, entails the convergence of the semigroups (in a particularly strong form).

The central result of this paper is an extension of the Trotter-Kato theorem to the quantum stochastic differential equations (QSDEs) defined in equations (1) and (2). The proof of the following theorem can be found in section 4.

Theorem 2 (QSDE Trotter-Kato). *The following conditions are equivalent.*

(a) *For every $\alpha, \beta \in \mathbb{C}^n$ and $u \in \mathcal{D}_0$, there exist $u^{(k)} \in \mathrm{Dom}(\mathscr{L}^{(k;\alpha\beta)})$ such that*

$$u^{(k)} \xrightarrow{k\to\infty} u, \qquad \mathscr{L}^{(k;\alpha\beta)} u^{(k)} \xrightarrow{k\to\infty} \mathscr{L}^{(\alpha\beta)} u.$$

(b) *For every $\psi \in \mathcal{H}_0 \otimes \mathcal{F}$ and $T < \infty$*

$$\lim_{k\to\infty} \sup_{0 \le t \le T} \|U_t^{(k)*}\psi - U_t^*\psi\| = 0.$$

Moreover, if $\mathcal{D}$ is a core for all $\mathscr{L}^{(k;\alpha\beta)}$, $k \in \mathbb{N}$ then we can always choose $\{u^{(k)}\} \subset \mathcal{D}$.

The convergence of $T_t^{(k;\alpha\beta)}$ to $T_t^{(\alpha\beta)}$ for all $\alpha, \beta \in \mathbb{C}^n$ thus turns out to be equivalent to the convergence, in a very strong sense, of $U_t^{(k)}$ to U_t. This is a powerful technique precisely because of the fact that convergence can be determined directly from the generators $\mathscr{L}^{(k;\alpha\beta)}$, $\mathscr{L}^{(\alpha\beta)}$ which depend only on the coefficients of equations (1) and (2); the technique does not require us to understand precisely the way in which $U_t^{(k)}$ depends on k. The benefit of this approach is clearly demonstrated by the proof of the singular perturbation results below, which follows as an application of theorem 2.

Remark 4. Note that for the implication (a)$\Rightarrow$(b) it always suffices to choose $\{u^{(k)}\} \subset \mathcal{D}$, as $\mathcal{D} \subset \mathrm{Dom}(\mathscr{L}^{(k;\alpha\beta)})$. It is only when we wish to ensure the existence of $\{u^{(k)}\} \subset \mathcal{D}$ in the reverse implication that we must assume $\mathcal{D}$ to be a core for all $\mathscr{L}^{(k;\alpha\beta)}$, $k \in \mathbb{N}$. In practice we are almost always interested in (a)$\Rightarrow$(b), where this is irrelevant.

Remark 5. The proof of theorem 2 is easily modified to show that $\|U_t^{(k)}\psi - U_t\psi\| \xrightarrow{k\to\infty} 0$ for every $\psi \in \mathcal{H}_0 \otimes \mathcal{F}$ and $t < \infty$. However, uniform convergence on compact time intervals appears to be easier to obtain for convergence to the adjoint U_t^*. As it is U_t^* which defines the physical time evolution, we do not attempt to obtain uniform convergence to U_t.

2.2. **Singular perturbations.** Let us start by introducing the basic assumptions. We consider $\mathcal{H}$ and $\mathcal{D}$ as fixed from the outset; as we will shortly see, we must choose $\mathcal{H}_0 \subset \mathcal{H}$ in a particular way.

Assumption 1 (Singular scaling). *There exist operators Y, $Y^\dagger$, A, $A^\dagger$, B, $B^\dagger$, F_i, $F_i^\dagger$, G_i, $G_i^\dagger$, W_{ij}, $W_{ij}^\dagger$ with the common invariant domain $\mathcal{D}$ such that*

$$K^{(k)} = k^2 Y + kA + B, \qquad L_i^{(k)} = kF_i + G_i, \qquad N_{ij}^{(k)} = W_{ij}$$

for all $k \in \mathbb{N}$ and $1 \le i, j \le n$.

Remark 6. Note that condition 1 imposes additional assumptions which are presumed to be in force throughout. In particular, the Hudson-Parthasarathy relations in condition 1 determine completely the form of the fourth coefficient $M_i^{(k)}$.

The interpretation of this scaling is that the dynamics of the physical system contains components that evolve on a fast time scale and on a slow time scale. The dynamics on the fast time scale is generated by quadratic ($\propto k^2$) term, while the coupling between the fast and slow time scales is determined by the linear ($\propto k$) term. As $k \to \infty$, the dynamical evolution of the fast components of the system will thus become increasingly singular. However, under appropriate conditions, the dynamical evolution of the slow components of the system will converge as $k \to \infty$, and the time evolution of these components in the limit will be decoupled from the fast components. Thus, in the limit of infinite separation of time scales, we have 'adiabatically eliminated' the fast components of the system leaving only an effective time evolution of the slow components. Typically the slow components of the system are the physical quantities of interest, and the elimination of the fast components from the model constitutes a significant simplification in many complex physical models.

As only slow dynamics should remain in the limit, we aim to choose $\mathcal{H}_0$ such that it contains only the slow degrees of freedom of the system. In order to ensure that the slow dynamics do indeed have a well defined limit, we must impose suitable structural assumptions on the coefficients; in particular, we must enforce that the terms which are linear and quadratic in k do not directly generate dynamics on $\mathcal{H}_0$—in the latter case, the limit would surely be singular on $\mathcal{H}_0$. We presently collect all the necessary structural assumptions; their significance can be clearly seen in the proof of our main result.

Assumption 2 (Structural requirements). *There is a closed subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that*

(a) *$P_0 \mathcal{D} \subset \mathcal{D}$;*
(b) *$Y P_0 = 0$ on $\mathcal{D}$;*
(c) *There exist $\tilde{Y}$, $\tilde{Y}^\dagger$ with the common invariant domain $\mathcal{D}$, so that $\tilde{Y}Y = Y\tilde{Y} = P_1$;*
(d) *$F_j^\dagger P_0 = 0$ on $\mathcal{D}$ for all $1 \le j \le n$;*
(e) *$P_0 A P_0 = 0$ on $\mathcal{D}$.*

Here P_0 denotes the orthogonal projection onto $\mathcal{H}_0$ and P_1 the orthogonal projection onto $\mathcal{H}_0^\perp$. We choose the dense domain $\mathcal{D}_0 = P_0 \mathcal{D}$ in $\mathcal{H}_0$.

It remains to introduce the limit coefficients. The following expressions emerge naturally in the proof of our main result through the application of theorem 2.

Assumption 3 (Limit coefficients). *Define the operators on $\mathcal{H}_0$*

$$K = P_0(B - A\tilde{Y}A)P_0, \qquad L_i = P_0(G_i - A\tilde{Y}F_i)P_0,$$

$$M_i = -\sum_{j=1}^{n} P_0 W_{ij}(G_j^\dagger - F_j^\dagger \tilde{Y} A)P_0, \qquad N_{ij} = \sum_{\ell=1}^{n} P_0 W_{i\ell}(F_\ell^\dagger \tilde{Y} F_j + \delta_{\ell j})P_0.$$

Then K, $K^\dagger$, L_i, $L_i^\dagger$, M_i, $M_i^\dagger$, N_{ij}, $N_{ij}^\dagger$ have common invariant domain $\mathcal{D}_0$. To ensure that these coefficients satisfy the Hudson-Parthasarathy relations of condition 1, we require

$$P_0(G_i - A\tilde{Y}F_i)P_1 = \sum_{\ell=1}^{n} P_0 W_{i\ell}(F_\ell^\dagger \tilde{Y} F_j + \delta_{\ell j})P_1 = \sum_{\ell=1}^{n} P_1 W_{i\ell}(F_\ell^\dagger \tilde{Y} F_j + \delta_{\ell j})P_0 = 0.$$

Our first order of business is to verify that the coefficients K, L_i, M_i, N_{ij} do indeed satisfy the Hudson-Parthasarathy relations in condition 1. This is proved in section 5.

Lemma 2. *Under assumptions 1–3, the coefficients K, L_i, M_i, N_{ij} satisfy the Hudson-Parthasarathy relations in condition 1.*

Finally, let us state our main result, whose proof will also be given in section 5. We remind the reader that beside the above assumptions, conditions 1–3 are still presumed to be in force; in particular, one must verify separately that (1) and (2) uniquely define contraction and unitary cocycles and that $\mathcal{D}_0$ is a core for $\mathscr{L}^{(\alpha\beta)}$, $\alpha, \beta \in \mathbb{C}^n$.

Theorem 3 (Singular perturbation). *Under assumptions 1–3, the singularly perturbed equations (1) converge to the limiting equation (2) on $\mathcal{H}_0$:*

$$\lim_{k\to\infty} \sup_{0 \le t \le T} \|U_t^{(k)*}\psi - U_t^*\psi\| = 0 \qquad \textit{for all } \psi \in \mathcal{H}_0 \otimes \mathcal{F}.$$

3. Examples

3.1. **Atom-cavity models with bounded coupling operators.** In this subsection, we will consider a class of physical models which consist of a harmonic oscillator coupled to auxiliary degrees of freedom. Such models cover various applications in quantum optics, such as a single mode optical cavity coupled to the internal degrees of freedom of a collection of atoms. Set $\mathcal{H} = \mathcal{H}' \otimes \ell^2(\mathbb{Z}_+)$. On the canonical basis $\{\varphi_i : i \in \mathbb{Z}_+\}$ of $\ell^2(\mathbb{Z}_+)$, the creation, annihilation and number operators are defined as

$$b^\dagger \varphi_i = \sqrt{i+1}\, \varphi_{i+1}, \qquad b\,\varphi_i = \sqrt{i}\,\varphi_{i-1}, \qquad b^\dagger b\,\varphi_i = i\,\varphi_i.$$

These definitions extend directly to the domain $\mathcal{D}' = \mathrm{span}\{\varphi_i : i \in \mathbb{Z}_+\} \subset \ell^2(\mathbb{Z}_+)$, and we will choose the dense domain $\mathcal{D} = \mathcal{H}' \overline{\otimes} \mathcal{D}'$ in $\mathcal{H}$. We now consider prelimit equations (1) whose coefficients have the following form on $\mathcal{D}$:

$$N_{ij}^{(k)} = S_{ij}^{(k)}, \qquad L_i^{(k)} = F_i^{(k)}\, b^\dagger + G_i^{(k)}, \qquad K^{(k)} = E_{11}^{(k)}\, b^\dagger b + E_{10}^{(k)}\, b^\dagger + E_{01}^{(k)}\, b + E_{00}^{(k)}.$$

Here $E_{pq}^{(k)}$, $F_i^{(k)}$, $G_i^{(k)}$, $S_{ij}^{(k)}$ are bounded operators on $\mathcal{H}'$, and condition 1 is presumed to be in force (in particular, $M_i^{(k)}$ are completely determined by these definitions).

We begin by showing that such equations possess unique solutions which extend to unitary cocycles on $\mathcal{H} \otimes \mathcal{F}$, and even that $\mathcal{D}$ is a core for $\mathscr{L}^{(k;\alpha\beta)}$, $\alpha, \beta \in \mathbb{C}$. We will establish these facts by verifying the conditions of Fagnola, see remark 3. Indeed, this is only a minor modification of the computations performed in [10, 12], and we refer to those papers for a more detailed account of the necessary steps.

Lemma 3. *The conditions of Fagnola for the existence of unique unitary cocycle solutions (remark 3) are satisfied for the class of models considered in this subsection.*

Proof. Note that all coefficients are linear combinations of operators of the form $X\, b^\dagger b$, $X\, b^\dagger$, $X\, b$, X, where X are bounded operators on $\mathcal{H}'$ whose norms are bounded by a fixed constant $\|X\| \le K < \infty$ (for simplicity, choose $K > 1$). Arguing as in [10, 12], we find

$$\|X(1)\cdots X(\ell)\,\psi \otimes \varphi_p\| \le \rho^\ell \sqrt{(p+\ell+m)!},$$

where $\psi \in \mathcal{H}'$, $X(1), \ldots X(\ell)$ is an arbitrary selection of coefficients or their adjoints, m is the number of occurences of $K^{(k)}$ or $K^{(k)\dagger}$, and $\rho = 4K^2$. Using the elementary estimate $(\ell + m)! \leq 2^{\ell+m} \ell! m!$, we can thus choose (see remark 3 for notations)

$$c(\ell, \psi \otimes \varphi_p) = \rho^\ell \, 2^\ell \sqrt{p! \, 2^p},$$

which does indeed satisfy the necessary analyticity requirement for sufficiently small ε. But any vector $u \in \mathcal{D}$ is the linear combination of a finite number of vectors of the form $\psi \otimes \varphi_p$, so that the claim follows easily. $\qquad\square$

With the existence and uniqueness taken care of, it becomes straightforward to apply our main result on singular perturbations, theorem 3. We will demonstrate this procedure in two physical applications: the adiabatic elimination of a single mode cavity, which extends the results of Gough and Van Handel [15], and the simultaneous elimination of a single mode cavity and the excited level of an atom as considered in Duan and Kimble [8].

Example 1 (Elimination of the cavity)**.** We consider the case where the coupling of the oscillator with the external field becomes increasingly strong; in the limit, we expect to eliminate the oscillator entirely from the model as it will be forced into its ground state. In quantum optics, this corresponds to the 'adiabatic elimination' of an optical cavity in the strong damping limit. To make this precise, we set

$$S_{ij}^{(k)} = S_{ij}, \quad F_i^{(k)} = kF_i, \quad G_i^{(k)} = G_i,$$
$$E_{11}^{(k)} = k^2 E_{11}, \quad E_{10}^{(k)} = kE_{10}, \quad E_{01}^{(k)} = kE_{01}, \quad E_{00}^{(k)} = E_{00}$$

for $1 \leq i, j \leq n$, where where S_{ij}, F_i, G_i, E_{pq} are bounded operators on $\mathcal{H}'$ which are chosen such that the Hudson-Parthasarathy relation of condition 1 are satisfied for all $k \in \mathcal{N}$. We obtain the following result, which does not presume conditions 2 and 3.

Proposition 1. *Suppose that E_{11} has a bounded inverse. Set $\mathcal{H}_0 = \mathcal{D}_0 = \mathcal{H}' \otimes \mathbb{C}\varphi_0$,*

$$K = E_{00} - E_{01}(E_{11})^{-1}E_{10}, \qquad L_i = G_i - E_{01}(E_{11})^{-1}F_i,$$
$$N_{ij} = \sum_{\ell=1}^n S_{i\ell}\{F_\ell^*(E_{11})^{-1}F_j + \delta_{\ell j}\}$$

on $\mathcal{D}_0$, and define M_i through the Hudson-Parthasarathy condition 1. Then

$$\lim_{k \to \infty} \sup_{0 \leq t \leq T} \|U_t^{(k)*}\psi - U_t^*\psi\| = 0 \qquad \text{for all } \psi \in \mathcal{H}_0 \otimes \mathcal{F}.$$

Proof. Note that condition 1 is satisfied by assumption for the prelimit equations (1), while condition 2 is satisfied for the prelimit equations by the Fagnola conditions verified above. Note that the singular scaling assumption 1 is also satisfied by construction. Next, we turn to the structural requirements of assumption 2. It is immediate that $\mathcal{D}_0 = P_0 \mathcal{D} \subset \mathcal{D}$, while $Y = E_{11} b^\dagger b$ evidently vanishes on $\mathcal{D}_0$. We define $\tilde{Y}$ on $\mathcal{D}$ by setting $\tilde{Y} \psi \otimes \varphi_0 = 0$ and $\tilde{Y} \psi \otimes \varphi_p = p^{-1}(E_{11})^{-1}\psi \otimes \varphi_p$ for $p \geq 1$; then indeed $\tilde{Y}Y = Y\tilde{Y} = P_1$ on $\mathcal{D}$. It is immediate that $F_j^\dagger b$ vanishes on $\mathcal{D}_0$ and that $A = E_{10} b^\dagger + E_{01} b$ satisfies $P_0 A P_0 = 0$. Hence assumption 2 is satisfied. Next we note that, taking into account the identity $X'b\tilde{Y}b^\dagger X \psi \otimes \varphi_0 = X'(E_{11})^{-1}X \psi \otimes \varphi_0$ for any bounded operators X, X' on $\mathcal{H}'$, the coefficients defined in the proposition coincide precisely with the coefficients in assumption 3, and it is easily verified that

the remaining conditions of assumption 3 are also satisfied. In particular, the limit coefficients satisfy the requirements of condition 1. But as the limit coefficients are bounded, conditions 2 and 3 are automatically satisfied. We have thus verified all the requirements of theorem 3. □

Example 2 (Duan-Kimble). We consider a three level atom coupled to a cavity which is itself coupled to a single output field, i.e., we set $\mathcal{H}' = \mathbb{C}^3$ and $n = 1$. Let us denote by $(|e\rangle, |+\rangle, |-\rangle)$ the canonical basis in $\mathbb{C}^3$. In this basis we define

$$\sigma_+^{(+)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \sigma_+^{(-)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define moreover $\sigma_-^{(\pm)} = \sigma_+^{(\pm)*}$ and $P_\pm = \sigma_-^{(\pm)}\sigma_+^{(\pm)}$. The quantum stochastic differential equation for a lambda system with one leg resonantly coupled to the cavity, under the rotating wave approximation and in the rotating frame, has the following coefficients:

$$S_{11}^{(k)} = I, \qquad G_1^{(k)} = 0, \qquad F_1^{(k)} = k\sqrt{\gamma}, \qquad E_{11}^{(k)} = -\frac{\gamma}{2}\,k^2,$$

$$E_{10}^{(k)} = -E_{01}^{(k)*} = k^2 g\sigma_-^{(+)}, \qquad E_{00}^{(k)} = k(\sigma_-^{(-)}\alpha^* - \sigma_+^{(-)}\alpha),$$

where $\gamma \in \mathbb{R}_+$ and $\alpha \in \mathbb{C}$. This model coincides with that of Duan and Kimble [8], except that we have allowed for additional driving on the uncoupled leg of the atom with amplitude α. Let us now define operators Y, A, B, F, G, W as

$$Y = -\frac{\gamma}{2}\,b^\dagger b + g\,(\sigma_-^{(+)}b^\dagger - \sigma_+^{(+)}b), \qquad A = \sigma_-^{(-)}\alpha^* - \sigma_+^{(-)}\alpha,$$

$$B = 0, \qquad F_1 = \sqrt{\gamma}\,b^\dagger, \qquad G_1 = 0, \qquad W_{11} = I.$$

These definitions are easily verified to satisfy condition 1 and assumption 1.

We proceed to verify assumption 2. Set $\mathcal{H}_0 = \mathcal{D}_0 = \text{span}\{|+\rangle \otimes \varphi_0, |-\rangle \otimes \varphi_0\}$. Then it is easily verified that $\mathcal{D}_0 = P_0\mathcal{D} \subset \mathcal{D}$ and that $YP_0 = F_1^\dagger P_0 = P_0AP_0 = 0$ on $\mathcal{D}$. It remains to define $\tilde{Y}$. To this end, define the following subspaces of $\mathcal{H}$:

$$\mathcal{H}_j = \text{span}\{|+\rangle \otimes \varphi_j, \ |-\rangle \otimes \varphi_j, \ |e\rangle \otimes \varphi_{j-1}\}, \qquad j \in \mathbb{N}.$$

Note that $P_1\mathcal{D} = \bigoplus_{j=1}^\infty \mathcal{H}_j$ and that the subspaces $\mathcal{H}_j, j \in \mathbb{N}$ are all invariant under the action of Y where, with respect to the basis $(|+\rangle \otimes \varphi_j, |-\rangle \otimes \varphi_j, |e\rangle \otimes \varphi_{j-1})$,

$$Y|_{\mathcal{H}_j} = \begin{pmatrix} -\frac{\gamma j}{2} & 0 & g\sqrt{j} \\ 0 & -\frac{\gamma j}{2} & 0 \\ -g\sqrt{j} & 0 & -\frac{\gamma(j-1)}{2} \end{pmatrix},$$

We may then construct $\tilde{Y} = \bigoplus_{j=1}^\infty (\tilde{Y}|_{\mathcal{H}_j})^{-1}$ on $P_1\mathcal{D}$ (and $\tilde{Y}\mathcal{D}_0 = 0$), which reads

$$\tilde{Y}|_{\mathcal{H}_j} = -\frac{1}{d_j}\begin{pmatrix} \frac{\gamma(j-1)}{2} & 0 & g\sqrt{j} \\ 0 & \frac{2d_j}{\gamma j} & 0 \\ -g\sqrt{j} & 0 & \frac{\gamma j}{2} \end{pmatrix}, \qquad d_j = \frac{\gamma^2 j(j-1)}{4} + g^2 j$$

in the same basis as above. Thus assumption 2 has been verified. We are now in a position to introduce the limit coefficients. By assumption 3, we must evidently define on $\mathcal{H}_0$

$$K = -\frac{|\alpha|^2\gamma}{2g^2}P_-, \qquad L_1 = -\frac{\alpha^*\sqrt{\gamma}}{g}\sigma_-^{(-)}\sigma_+^{(+)}, \qquad N_{11} = I - 2P_-.$$

It is again easily verified by explicit computation that the remaining conditions of assumption 3 are satisfied. Moreover, all the coefficients that we have introduced satisfy the Hudson-Parthasarathy relations of condition 1, condition 2 is satisfied for the prelimit equations by the Fagnola conditions, and conditions 1 and 3 are automatically satisfied as the limit coefficients are bounded. Thus all the conditions of theorem 3 have been verified.

3.2. **Coupled oscillators.** In the examples in the previous subsection, the application of theorem 3 was significantly simplified by two convenient properties: the Fagnola conditions could be verified for that class of models, and the limit equations always had bounded coefficients. In the present section, we will develop an example which enjoys neither of these properties.

We consider a single mode cavity in which one of the mirrors is allowed to oscillate along the cavity axis (see, e.g., [7]). The initial system will therefore have two degrees of freedom: the kinematic observables of the oscillating mirror and the usual observables associated with the cavity mode. This is implemented by choosing $\mathcal{H} = L^2(\mathbb{R}) \otimes \ell^2(\mathbb{Z}_+)$. For the cavity mode Hilbert space $\ell^2(\mathbb{Z}_+)$, we define the domain $\mathcal{D}'$ of finite particle vectors and the creation and annihilation operators $b^\dagger$, b as in the previous section. In the mirror Hilbert space $L^2(\mathbb{R})$ we will choose as dense domain the Schwartz space [13, ch. 8–9]

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}) : \|X^\alpha D^\beta f\|_\infty < \infty \ \forall \alpha, \beta \in \mathbb{Z}_+\},$$

where we have defined the operators

$$Df(x) = \frac{df(x)}{dx}, \qquad Xf(x) = xf(x) \qquad \forall f \in C^\infty(\mathbb{R}).$$

Note that these operators leave $\mathcal{S}$ invariant, as do all operators of the following form:

$$(g(X)f)(x) = g(x)f(x) \qquad \forall f \in \mathcal{S}, \ g \in C^\infty(\mathbb{R}) \text{ s.t. } \|D^\beta g\|_\infty < \infty \text{ for all } \beta \in \mathbb{Z}_+.$$

In addition, let us recall the following well known facts [23, appendix V.3]. If we define the operators $B, B^\dagger$ with invariant domain $\mathcal{S}$ and the vectors $\Phi_j \in \mathcal{S}$ as

$$B = \frac{X + D}{\sqrt{2}}, \qquad B^\dagger = \frac{X - D}{\sqrt{2}}, \qquad \Phi_j = \pi^{-1/4}(j!)^{-1/2}(B^\dagger)^j e^{-x^2/2}, \ j \in \mathbb{Z}_+,$$

then $\mathcal{S}' = \text{span}\{\Phi_j : j \in \mathbb{Z}_+\} \subset \mathcal{S}$ is dense in $L^2(\mathbb{R})$, $B^\dagger B \Phi_j = j\Phi_j$, $B\Phi_j = \sqrt{j}\,\Phi_{j-1}$, and $B^\dagger \Phi_j = \sqrt{j+1}\,\Phi_{j+1}$. Therefore $\mathcal{S}'$ corresponds to the domain of finite particle vectors in $\ell^2(\mathbb{Z}_+)$ in the usual isomorphism between $L^2(\mathbb{R})$ and $\ell^2(\mathbb{Z}_+)$. In the following we will set $\mathcal{D} = \mathcal{S} \,\overline{\otimes}\, \mathcal{D}'$, but we will occasionally exploit $\mathcal{S}'$ whenever this is convenient.

We are now in a position to introduce the prelimit equations (1). We set $n = 1$, i.e., there is only one external field, and choose the following prelimit coefficients:

$$N_{11}^{(k)} = I, \qquad L_1^{(k)} = k\sqrt{\gamma}\, b^\dagger, \qquad K^{(k)} = k^2 \left[i\vartheta\,(B + B^\dagger) - \frac{\gamma}{2} \right] b^\dagger b + i\Omega\, B^\dagger B,$$

where $\vartheta, \gamma, \Omega > 0$ are fixed parameters. The physical interpretation of this model is that the optical frequency of the cavity mode is determined by the length of the cavity, and hence by the displacement $(B + B^\dagger)$ of the mirror; this accounts for the $(B + B^\dagger)b^\dagger b$ term. At the same time, we presume that the mirror is oscillating in a quadratic potential, which accounts for the $B^\dagger B$ term, while the remaining terms couple the cavity mode to the external field. In the strong damping limit $k \to \infty$

we expect as before that the cavity is eliminated, so that we obtain an effective interaction between the mirror and the external field.

Lemma 4. *The prelimit equations (1) each possess a unique solution which extends to a contraction cocycle on $\mathcal{H} \otimes \mathcal{F}$.*

Proof. First, note that condition 1 is satisfied for these equations on the domain $\mathcal{D}$. It suffices to verify that the closure of $\mathscr{L}^{(k;\alpha\beta)}$ on $\mathcal{D}$ is the generator of a strongly continuous contraction semigroup for every $k \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{C}$; see remark 3 and [20, theorem 1.1]. Fix k, α, β, and denote by $\mathscr{L}$ the operator $\mathscr{L}^{(k;\alpha\beta)}$ with the domain $\mathcal{D}$. By an easy corollary of the Lumer-Phillips theorem [20, lemma 2.1], it suffices to prove that $\mathcal{D}$ is a core for $\mathscr{L}^*$. But this follows immediately from the following lemma, and the proof is complete. $\qquad\square$

Lemma 5. *Let T be an operator on the domain $\mathcal{D}$ of the following form:*

$$T = c_1 + c_2 b^\dagger b + c_3 B^\dagger B + c_4 b + c_5 b^\dagger + c_6 (B + B^\dagger) b^\dagger b,$$

where $c_{1,\dots,6} \in \mathbb{C}$. Then $\mathcal{D}$ is a core for T^*.

Proof. As $\langle u, Tv \rangle = \langle T^\dagger u, v \rangle$ for $u, v \in \mathcal{D}$, clearly $\mathcal{D} \subset \mathrm{Dom}(T^*)$ where $T^*|_{\mathcal{D}} = T^\dagger$. Our goal is to show that the closure $\mathrm{cl}\, T^\dagger$ of $T^\dagger$ is in fact T^*, i.e., we must show that $\mathrm{Dom}(T^*) \subset \mathrm{Dom}(\mathrm{cl}\, T^\dagger)$. We begin by noting that

$$\mathrm{Dom}(\mathrm{cl}\, T^\dagger) = \left\{ \sum_{j,\ell=0}^\infty d_{j,\ell}\, \Phi_j \otimes \varphi_\ell : \sum_{j,\ell=0}^\infty |d_{j,\ell}|^2 < \infty \text{ and } \sum_{j,\ell=0}^\infty |T_{j,\ell}(d)|^2 < \infty \right\},$$

where we have written

$$T_{j,\ell}(d) = (c_1^* + c_2^* \ell + c_3^* j) d_{j,\ell} + c_4^* \sqrt{\ell}\, d_{j,\ell-1}$$
$$+ c_5^* \sqrt{\ell+1}\, d_{j,\ell+1} + c_6^* (\sqrt{j+1}\, d_{j+1,\ell} + \sqrt{j}\, d_{j-1,\ell}) \ell.$$

Indeed, this space is evidently the completion of $\mathcal{S}' \overline{\otimes} \mathcal{D}'$ with respect to the Sobolev norm $\|v\|_{T^\dagger} = \|v\| + \|T^\dagger v\|$, and $\mathcal{D}$ is already included in this space (this follows from the representation [23, theorem V.13]). It thus remains to show that $\mathrm{Dom}(T^*)$ is included in this space as well. To this end, let us define for all $N \in \mathbb{N}$

$$v = \sum_{j,\ell=0}^\infty d_{j,\ell}\, \Phi_j \otimes \varphi_\ell, \qquad \sum_{j,\ell=0}^\infty |d_{j,\ell}|^2 < \infty, \qquad u_N = \frac{\sum_{j,\ell=0}^N T_{j,\ell}(d)\, \Phi_j \otimes \varphi_\ell}{\| \sum_{j,\ell=0}^N T_{j,\ell}(d)\, \Phi_j \otimes \varphi_\ell \|}.$$

Thus $\|u_N\| = 1$ and $u_N \in \mathcal{D}$ for all $N \in \mathbb{N}$. Now recall that by the Riesz representation theorem $v \in \mathrm{Dom}(T^*)$ iff $|\langle Tu, v \rangle| \leq C\|u\|$ for all $u \in \mathcal{D}$. But if $v \notin \mathrm{Dom}(\mathrm{cl}\, T^\dagger)$, then

$$|\langle Tu_N, v \rangle| = |\langle Tu_N, P_{N+1}v \rangle| = |\langle u_N, T^\dagger P_{N+1}v \rangle| = \left\| \sum_{j,\ell=0}^N T_{j,\ell}(d)\, \Phi_j \otimes \varphi_\ell \right\| \xrightarrow{N\to\infty} \infty$$

where P_N is the orthogonal projection onto the subspace $\mathrm{span}\{\Phi_j \otimes \varphi_\ell : j, \ell \leq N\}$. Thus $v \notin \mathrm{Dom}(\mathrm{cl}\, T^\dagger)$ implies $v \notin \mathrm{Dom}(T^*)$, so that $\mathrm{Dom}(T^*) \subset \mathrm{Dom}(\mathrm{cl}\, T^\dagger)$ as desired. $\qquad\square$

Let us now turn to the assumptions 1–3. Clearly our coefficients are of the form required by assumption 1. Let us set $\mathcal{H}_0 = L^2(\mathbb{R}) \otimes \mathbb{C}\varphi_0$ and $\mathcal{D}_0 = P_0 \mathcal{D} = \mathcal{S} \overline{\otimes} \mathbb{C}\varphi_0$.

Then all the requirements of assumption 2 are clearly satisfied except that we must verify that

$$Y = \left[i\vartheta\,(B + B^\dagger) - \frac{\gamma}{2}\right] b^\dagger b$$

possesses an inverse $\tilde{Y}$ on $P_1 \mathcal{D}$. We may simply set, however,

$$\tilde{Y}\,\psi \otimes \varphi_0 = 0, \qquad \tilde{Y}\,\psi \otimes \varphi_j = j^{-1}\left[i\vartheta\,(B + B^\dagger) - \frac{\gamma}{2}\right]^{-1} \psi \otimes \varphi_j, \quad j \geq 1.$$

As $B + B^\dagger = X\sqrt{2}$ and the function $g : \mathbb{R} \to \mathbb{C}$, $g(x) = (i\vartheta x\sqrt{2} - \gamma/2)^{-1}$ is smooth and bounded together with all its derivatives, the bounded operator $\tilde{Y}$ leaves the Schwartz space invariant, and clearly $Y\tilde{Y} = \tilde{Y}Y = P_1$ on $\mathcal{D}$. Therefore, all the conditions of assumption 2 are satisfied. It should be noted that it would not have been sufficient to choose the smaller finite excitation domain $\mathcal{S}'$ in $L^2(\mathbb{R})$, despite the fact that it is left invariant by all prelimit coefficients, as the inverse $\tilde{Y}$ does not leave $\mathcal{S}'$ invariant.

Having verified assumptions 1 and 2, the form of the limit equation (2) is determined by assumption 3. In particular, we must choose the limit coefficients

$$N_{11} = \frac{i\vartheta\,(B + B^\dagger) + \gamma/2}{i\vartheta\,(B + B^\dagger) - \gamma/2}, \qquad L_1 = 0, \qquad K = i\Omega\,B^\dagger B$$

on $\mathcal{D}_0$, and one can verify by straightforward manipulation that the remaining conditions in assumption 3 are satisfied. However, we have not yet established that the limit equation possesses a unique solution which extends to a unitary cocycle, and whether the core condition 3 is satisfied. This is our next order of business.

Lemma 6. *The limit equation (2) possesses a unique solution which extends to a contraction cocycle on $\mathcal{H}_0 \otimes \mathcal{F}$. Moreover, the core condition 3 is satisfied.*

Proof. First, note that condition 1 is satisfied for the limit equation on the domain $\mathcal{D}_0$. It suffices to verify that the closure of $\mathscr{L}^{(\alpha\beta)}$ on $\mathcal{D}_0$ is the generator of a strongly continuous contraction semigroup for every $\alpha, \beta \in \mathbb{C}$: existence and uniqueness follows from remark 3 and [20, theorem 1.1], while the fact that $\mathcal{D}_0$ is a core for the corresponding generators is then trivially satisfied. Note that we have

$$\mathscr{L}^{(\alpha\beta)} = \alpha^*\beta\,\frac{i\vartheta\,(B + B^\dagger) + \gamma/2}{i\vartheta\,(B + B^\dagger) - \gamma/2} - \frac{|\alpha|^2 + |\beta|^2}{2} + i\Omega\,B^\dagger B.$$

But as $B^\dagger B$ is essentially self-adjoint on $\mathcal{S}$, the closure of $i\Omega\,B^\dagger B$ on $\mathcal{S}$ is the generator of a strongly continuous unitary group (and hence of a contraction semigroup), while the first two terms in the expression for $\mathscr{L}^{(\alpha\beta)}$ constitute a bounded and dissipative perturbation. The result therefore follows from a standard perturbation argument [5, theorem 3.7]. $\qquad\square$

Lemma 7. *The solution of equation (2) extends to a unitary cocycle.*

Proof. As we have shown that there is a unique solution, it suffices to show that there exists a unitary process U_t that satisfies equation (2). To this end, consider the equations

$$S_t = I + \int_0^t S_s\,i\Omega\,B^\dagger B\,ds, \qquad R_t = I + \int_0^t R_s\,(S_s N_{11} S_s^\dagger - I)\,d\Lambda_s.$$

Then S_t is the strongly continuous unitary group generated by the closure of $i\Omega B^\dagger B$, while R_t satisfies a Hudson-Parthasarathy equation with bounded, though time

dependent, coefficients. The latter possesses a unitary solution as can be verified, e.g., using the results in [17]. Moreover, using the quantum stochastic calculus, it is immediately verified that $U_t = R_t S_t$ satisfies equation (2), and $R_t S_t$ is clearly unitary. $\qquad\square$

We have finally verified all the conditions 1–3 and assumptions 1–3. We can thus invoke theorem 3, and we find that indeed

$$\lim_{k\to\infty}\sup_{0\le t\le T}\|U_t^{(k)*}\psi - U_t^*\psi\| = 0 \qquad \text{for all } \psi \in \mathcal{H}_0 \otimes \mathcal{F}$$

with the prelimit and limit coefficients as define above.

3.3. **Finite dimensional approximations.** Suppose that we are given a quantum stochastic differential equation of the form (2) with $\dim \mathcal{H}_0 = \infty$. Though such an equation may be a realistic physical model, it can not be simulated directly on a computer. To perform numerical computations (typically one would simulate a derivative of this equation, such as a master equation or a quantum filtering equation), we must first approximate this infinite dimensional equation by one which is finite dimensional. A very common way of doing this is to fix an orthonormal basis $\{\psi_\ell\}_{\ell\ge 0} \subset \mathcal{D}_0$, so that we can approximate the coefficients in the equation for U_t by their truncations with respect to the first k basis elements. We will show that the solutions of the truncated equations do in fact converge to U_t as $k \to \infty$. Though the result is of some interest in itself, it also serves as an exceedingly simple demonstration of the Trotter-Kato theorem 2 which is not of the singular perturbation type (theorem 3).

In the current setting, we will simply set $\mathcal{H}_0 = \mathcal{H}$. We presume that the equation (2) is given on the domain $\mathcal{D}_0 \,\overline{\otimes}\, \mathcal{E}$ and that it satisfies conditions 1–3. We also presume that $\mathcal{D}_0 = \mathrm{span}\{\psi_\ell : \ell \in \mathbb{Z}_+\}$, where $\{\psi_\ell\}_{\ell\ge 0}$ is an orthonormal basis of $\mathcal{H}$. For simplicity, let us assume that $N_{ij} = \delta_{ij}$; this is not essential, see remark 7 below.

Define $P^{(k)}$ to be the orthogonal projection onto $\mathrm{span}\{\psi_0, \ldots, \psi_k\}$. We proceed to define the equations (1) by truncating the coefficients. Since the truncated operators will be bounded, we set $\mathcal{D} = \mathcal{H}$ and we let $M_i^{(k)}, L_i^{(k)}$ and $K^{(k)}$ in equation (1) be given by

$$L_i^{(k)} = P^{(k)} L_i P^{(k)}, \qquad M_i^{(k)} = -L_i^{(k)*}, \qquad K^{(k)} = P^{(k)} K P^{(k)}$$

for all $1 \le i \le n$. Note that condition 1 is thus satisfied, and as the coefficients are bounded condition 2 is as well. Conditions 1–3 are satisfied for the limit equation by assumption.

Proposition 2 (Finite dimensional approximation)**.** *Under the above assumptions, the truncated equations (1) converge to the exact equation (2):*

$$\lim_{k\to\infty}\sup_{0\le t\le T}\|U_t^{(k)*}\psi - U_t^*\psi\| = 0 \qquad \text{for all } \psi \in \mathcal{H} \otimes \mathcal{F}.$$

Proof. By theorem 2, it suffices to show that for every $\alpha, \beta \in \mathbb{C}^n$ and $u \in \mathrm{span}\{\psi_\ell\}_{\ell\ge 0}$, there exists a sequence $\{u^{(k)}\} \subset \mathcal{H}$ such that

$$u^{(k)} \xrightarrow{\,k\to\infty\,} u, \qquad \mathscr{L}^{(k;\alpha\beta)} u^{(k)} \xrightarrow{\,k\to\infty\,} \mathscr{L}^{(\alpha\beta)} u.$$

We may simply take $u^{(k)} = P^{(k)}u$. Since u is an element of the linear span of $\{\psi_\ell\}_{\ell \geq 0}$, there is a $C \in \mathbb{N}$ such that $u^{(k)} = u$ and $\mathscr{L}^{(k;\alpha\beta)}u^{(k)} = \mathscr{L}^{(\alpha\beta)}u$ for all $k \geq C$. $\qquad\square$

Remark 7. The assumption that $N_{ij} = \delta_{ij}$ is not necessary for the result to hold; however, care must be taken to approximate N_{ij} in such a way that the Hudson-Parthasarathy relations of condition 2 remain satisfied (simple truncation as above typically does not achieve this purpose). Similarly, it is not difficult to show that the result still holds if only $\text{span}\{\psi_\ell : \ell \in \mathbb{Z}_+\} \subset \mathcal{D}_0$, provided that $\text{span}\{\psi_\ell : \ell \in \mathbb{Z}_+\}$ is a core for $\mathscr{L}^{(\alpha\beta)}$, $\alpha, \beta \in \mathbb{C}^n$. The chief technical advantage of this simple result compared to, e.g., an application of the results in [21], is the very strong nature of the convergence.

4. Proof of the main convergence theorem

Before turning to the proof of the main convergence theorem, we provide a proof of the semigroup properties of $T_t^{(\alpha\beta)}$, lemma 1.

Proof of Lemma 1. As U_t is a contraction cocycle, it follows easily that $T_t^{(\alpha\beta)}$ is a strongly continuous contraction semigroup. As $\alpha I_{[0,t]} \in \mathfrak{S}$ for every $\alpha \in \mathbb{C}^n$ and $t \geq 0$ (recall that we assume that $\mathfrak{S}$ contains all simple functions), we obtain from (1) for $u, v \in \mathcal{D}_0$

$$\langle u \otimes e(\alpha I_{[0,t]}), U_t\, v \otimes e(\beta I_{[0,t]}) \rangle =$$
$$e^{\alpha^*\beta\, t}\left[\langle u, v \rangle + \int_0^t e^{-\alpha^*\beta\, s} \langle u \otimes e(\alpha I_{[0,s]}), U_s\, (\overline{\mathscr{L}}^{(\alpha\beta)}v \otimes e(\beta I_{[0,s]})) \rangle \, ds \right],$$

where we have written for $u \in \mathcal{D}_0$

$$\overline{\mathscr{L}}^{(\alpha\beta)}u = \left(\sum_{i,j=1}^n \alpha_i^*(N_{ij} - \delta_{ij})\beta_j + \sum_{i=1}^n \alpha_i^* M_i + \sum_{i=1}^n L_i \beta_i + K \right) u.$$

Therefore, we obtain for $u, v \in \mathcal{D}_0$ using the chain rule

$$(4) \qquad\qquad \langle u, T_t^{(\alpha\beta)}v \rangle = \langle u, v \rangle + \int_0^t \langle u, T_s^{(\alpha\beta)} \mathscr{L}^{(\alpha\beta)}v \rangle \, ds,$$

with $\mathscr{L}^{(\alpha\beta)}v$ defined as in equation (3). Using the fact that $\mathcal{D}_0$ is dense in $\mathcal{H}_0$ and that

$$|\langle u, T_s^{(\alpha\beta)} \mathscr{L}^{(\alpha\beta)}v \rangle| \leq \|u\| \, \|T_s^{(\alpha\beta)} \mathscr{L}^{(\alpha\beta)}v\| \leq \|u\| \, \|\mathscr{L}^{(\alpha\beta)}v\|,$$

we find using dominated convergence that (4) holds identically for all $u \in \mathcal{H}_0$, $v \in \mathcal{D}_0$, and $t \geq 0$. Moreover, as $\|T_t^{(\alpha\beta)} \mathscr{L}^{(\alpha\beta)}v\| \leq \|\mathscr{L}^{(\alpha\beta)}v\|$ is bounded, it follows that $T_t^{(\alpha\beta)} \mathscr{L}^{(\alpha\beta)}v$ is Bochner integrable and thus evidently (see, e.g., [9, section 1.2])

$$T_t^{(\alpha\beta)}v = v + \int_0^t T_s^{(\alpha\beta)} \mathscr{L}^{(\alpha\beta)}v \, ds \qquad \forall\, t \geq 0, \; v \in \mathcal{D}_0.$$

But then we obtain using the strong continuity of $T_t^{(\alpha\beta)}$

$$\lim_{t \searrow 0} \frac{T_t^{(\alpha\beta)}v - v}{t} = \mathscr{L}^{(\alpha\beta)}v \qquad \forall\, v \in \mathcal{D}_0,$$

which establishes the claim. The result for $T_t^{(k;\alpha\beta)}$ follows identically. $\qquad\square$

The remainder of this section is devoted to the proof of theorem 2. We first prove weak convergence of $U_t^{(k)}$ to U_t for every $t < \infty$; this is the content of the following lemma. The weak convergence will subsequently be strengthened to strong convergence, and ultimately to strong convergence uniformly on compact time intervals.

Lemma 8. *If for every $\alpha, \beta \in \mathbb{C}^n$ and $u \in \mathcal{D}_0$, there exist $u^{(k)} \in \mathrm{Dom}(\mathscr{L}^{(k;\alpha\beta)})$ so that*

$$u^{(k)} \xrightarrow{k \to \infty} u, \qquad \mathscr{L}^{(k;\alpha\beta)} u^{(k)} \xrightarrow{k \to \infty} \mathscr{L}^{(\alpha\beta)} u,$$

then $\langle \psi_1, U_t^{(k)} \psi_2 \rangle \xrightarrow{k \to \infty} \langle \psi_1, U_t \psi_2 \rangle$ for every $\psi_1, \psi_2 \in \mathcal{H}_0 \otimes \mathcal{F}$ and $t < \infty$.

Proof. As U_t and $U_t^{(k)}$ are adapted, it suffices to restrict our attention to $\mathcal{H}_0 \otimes \mathcal{F}_{t]}$. Let $\mathcal{U} \subset \mathcal{H}_0 \otimes \mathcal{F}_{t]}$ be a total subset. Then for any $\psi_1, \psi_2 \in \mathcal{H}_0 \otimes \mathcal{F}_{t]}$ and $\varepsilon > 0$, there exist $\psi_1', \psi_2' \in \mathrm{span}\,\mathcal{U}$ such that $\|\psi_1 - \psi_1'\| < \varepsilon$, $\|\psi_2 - \psi_2'\| < \varepsilon$, and

$$\limsup_{k \to \infty} |\langle \psi_1, (U_t^{(k)} - U_t)\psi_2 \rangle|$$

$$\leq 2\|\psi_1 - \psi_1'\|\,\|\psi_2\| + 2\|\psi_2 - \psi_2'\|\,\|\psi_1'\| + \limsup_{k \to \infty} |\langle \psi_1', (U_t^{(k)} - U_t)\psi_2' \rangle|$$

$$\leq 2\varepsilon(\|\psi_1\| + \|\psi_2\|) + 2\varepsilon^2 + \limsup_{k \to \infty} |\langle \psi_1', (U_t^{(k)} - U_t)\psi_2' \rangle|.$$

It thus suffices to prove that the limit on the right vanishes for every $\psi_1', \psi_2' \in \mathrm{span}\,\mathcal{U}$, i.e., it suffices to prove the result for $\psi_1, \psi_2 \in \mathcal{U}$ only.

Now consider the total subset $\mathcal{U} = \{u \otimes e(f) : u \in \mathcal{D}_0, \ f \in \mathfrak{S}'\}$, where $\mathfrak{S}'$ is the set of simple functions in $L^2([0, t]; \mathbb{C}^n)$ (recall that by assumption $\mathfrak{S}'$ is admissible, so that $\mathcal{U} \subset \mathcal{D}_0 \overline{\otimes} \mathcal{E}_{t]}$). Let $\psi_i \in \mathcal{U}$, $i \in \{1, 2\}$. Then there exist $u_i \in \mathcal{D}_0$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = t$, and $\alpha_{i0}, \ldots, \alpha_{im} \in \mathbb{C}^n$ such that $\psi_i = u_i \otimes e(f_i)$ with $f_i(s) = \alpha_{ij}$ for $s \in [t_j, t_{j+1}[$. It is not difficult to verify that, by virtue of the cocycle property,

$$\langle \psi_1, U_t^{(k)} \psi_2 \rangle = \|e(f_1)\|\,\|e(f_2)\|\,\langle u_1, T_{t_1}^{(k;\alpha_{10}\alpha_{20})} T_{t_2-t_1}^{(k;\alpha_{11}\alpha_{21})} \cdots T_{t-t_m}^{(k;\alpha_{1j}\alpha_{2j})} u_2 \rangle,$$

and similarly for $\langle \psi_1, U_t \psi_2 \rangle$. In particular, the result follows as

$$|\langle \psi_1, (U_t^{(k)} - U_t)\psi_2 \rangle|$$

$$= \|e(f_1)\|\,\|e(f_2)\|\,|\langle u_1, (T_{t_1}^{(k;\alpha_{10}\alpha_{20})} \cdots T_{t-t_m}^{(k;\alpha_{1j}\alpha_{2j})} - T_{t_1}^{(\alpha_{10}\alpha_{20})} \cdots T_{t-t_m}^{(\alpha_{1j}\alpha_{2j})})u_2 \rangle|$$

$$\leq \|e(f_1)\|\,\|e(f_2)\|\,\|u_1\|\,\|(T_{t_1}^{(k;\alpha_{10}\alpha_{20})} \cdots T_{t-t_m}^{(k;\alpha_{1j}\alpha_{2j})} - T_{t_1}^{(\alpha_{10}\alpha_{20})} \cdots T_{t-t_m}^{(\alpha_{1j}\alpha_{2j})})u_2\|,$$

which converges to zero as $k \to \infty$ by the Trotter-Kato theorem. $\qquad \square$

Proof of Theorem 2 (a)⇒(b). Weak convergence of $U_t^{(k)}$ to U_t was proved in the preceding lemma. But note that as $U_t^{(k)}$ are contractions and U_t is unitary, we can write

$$\|(U_t^{(k)} - U_t)^*\psi\|^2 = \langle (U_t^{(k)} - U_t)^*\psi, (U_t^{(k)} - U_t)^*\psi \rangle \leq 2\|\psi\|^2 - 2\,\mathrm{Re}(\langle U_t^{(k)*}\psi, U_t^*\psi \rangle).$$

As $\langle \psi, U_t^{(k)} U_t^* \psi \rangle \to \|\psi\|^2$ as $k \to \infty$ follows from the previous lemma, we immediately obtain strong convergence. It thus remains to prove that strong convergence for every fixed time t can be strengthened to strong convergence uniformly on compact time intervals. To this end, we will appeal again to the Trotter-Kato theorem in a slightly different manner.

It is convenient to extend the Fock space to two-sided time, i.e., we will consider the ampliations of all our operators to the extended Fock space $\tilde{\mathcal{F}} = \Gamma_s(L^2(\mathbb{R}; \mathbb{C}^n)) \cong \mathcal{F}_- \otimes \mathcal{F}$, where $\mathcal{F}_- \cong \mathcal{F}$ is the negative time portion of the two-sided Fock space. We now define the two-sided shift $\tilde{\theta}_t : L^2(\mathbb{R}; \mathbb{C}^n) \to L^2(\mathbb{R}; \mathbb{C}^n)$ as $\tilde{\theta}_t f(s) = f(t+s)$, and by $\tilde{\Theta}_t : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}$ its second quantization. Note that $\tilde{\Theta}_t$ is a strongly continuous one-parameter unitary group, and that the cocycle property reads $U_{t+s} = U_s \tilde{\Theta}_s^* U_t \tilde{\Theta}_s$, etc., in terms of the two-sided shift. Now define on the two-sided Fock space the operators

$$V_t^{(k)} = \tilde{\Theta}_t U_t^{(k)*}, \qquad\qquad V_t = \tilde{\Theta}_t U_t^*.$$

Then it is immediate from the cocycle property that $V_t^{(k)}$ and V_t define strongly continuous contraction semigroups on $\mathcal{H} \otimes \tilde{\mathcal{F}}$ and $\mathcal{H}_0 \otimes \tilde{\mathcal{F}}$, respectively, whose generators we will denote as $\tilde{\mathscr{L}}^{(k)}$ and $\tilde{\mathscr{L}}$ (see [16] for a detailed study in the bounded case). Moreover,

$$\|U_t^{(k)*}\psi - U_t^*\psi\| = \frac{\|V_t^{(k)}\psi_- \otimes \psi - V_t \psi_- \otimes \psi\|}{\|\psi_-\|} \qquad \forall\, \psi \in \mathcal{H}_0 \otimes \mathcal{F},\ \psi_- \in \mathcal{F}_-$$

as $\tilde{\Theta}_t$ is an isometry, where $\psi_- \otimes \psi \in \mathcal{F}_- \otimes \mathcal{H}_0 \otimes \mathcal{F} \cong \mathcal{H}_0 \otimes \tilde{\mathcal{F}}$. Hence, by the Trotter-Kato theorem, it suffices to show the following: for any $\psi \in \mathrm{Dom}(\mathscr{L})$, there exist $\psi^{(k)} \in \mathrm{Dom}(\mathscr{L}^{(k)})$ such that $\psi^{(k)} \to \psi$ and $\mathscr{L}^{(k)}\psi^{(k)} \to \mathscr{L}\psi$ as $k \to \infty$. Let us prove this assertion. Fix $\psi \in \mathrm{Dom}(\mathscr{L})$ and $\lambda > 0$, and define

$$\psi^{(k)} = \int_0^\infty e^{-\lambda t} V_t^{(k)}(\lambda \psi - \mathscr{L}\psi)\, dt = R_\lambda^{(k)}(\lambda \psi - \mathscr{L}\psi),$$

where $R_\lambda^{(k)}$ is the resolvent of $V_t^{(k)}$. We have already shown that $\|U_t^{(k)*}\psi - U_t^*\psi\| \to 0$ as $k \to \infty$ for every fixed $t \geq 0$ and $\psi \in \mathcal{H}_0 \otimes \mathcal{F}$, so that evidently

$$\|V_t^{(k)}\psi - V_t\psi\| \xrightarrow{k \to \infty} 0 \qquad \forall\, \psi \in \mathcal{H}_0 \otimes \tilde{\mathcal{F}},\ t \geq 0.$$

Therefore, we find by dominated convergence that

$$\psi^{(k)} \xrightarrow{k \to \infty} \int_0^\infty e^{-\lambda t} V_t(\lambda \psi - \mathscr{L}\psi)\, dt = R_\lambda(\lambda \psi - \mathscr{L}\psi) = \psi,$$

where we have used a standard result on resolvents [5, section 2.1]. Similarly, we find that

$$\psi^{(k)} \in \mathrm{Dom}(\mathscr{L}^{(k)}), \qquad \mathscr{L}^{(k)}\psi^{(k)} = \mathscr{L}^{(k)}R_\lambda^{(k)}(\lambda \psi - \mathscr{L}\psi) = \mathscr{L}\psi + \lambda(\psi^{(k)} - \psi)$$

for every $k \in \mathbb{N}$ by virtue of another standard result on resolvents [5, *loc. cit.*]. Thus evidently $\mathscr{L}^{(k)}\psi^{(k)} \to \mathscr{L}\psi$ as $k \to \infty$, and the proof is complete. $\square$

Proof of Theorem 2 (b)$\Rightarrow$(a). Choose $\alpha, \beta \in \mathbb{C}^n$, $v \in \mathcal{H}_0$, and let $\psi = v \otimes e(\beta I_{[0,t]})$. We obtain an estimate on $\|(T_t^{(k;\alpha\beta)} - T_t^{(\alpha\beta)})v\|$ through the following steps:

$$\|(T_t^{(k;\alpha\beta)} - T_t^{(\alpha\beta)})v\| = \sup_{\substack{u \in \mathcal{H} \\ \|u\| \leq 1}} |\langle u, (T_t^{(k;\alpha\beta)} - T_t^{(\alpha\beta)})v\rangle| =$$

$$\sup_{\substack{u \in \mathcal{H} \\ \|u\| \leq 1}} \frac{|\langle u \otimes e(\alpha I_{[0,t]}), (U_t^{(k)} - U_t)\psi\rangle|}{e^{(|\alpha|^2 + |\beta|^2)t/2}} \leq \sup_{\substack{\psi' \in \mathcal{H} \otimes \mathcal{F} \\ \|\psi'\| \leq 1}} \frac{|\langle \psi', (U_t^{(k)} - U_t)\psi\rangle|}{e^{|\beta|^2 t/2}} =$$

$$e^{-|\beta|^2 t/2}\, \|(U_t^{(k)} - U_t)\psi\| \leq e^{-|\beta|^2 t/2}\sqrt{\|\psi\|^2 - 2\,\mathrm{Re}(\langle U_t^{(k)}\psi, U_t\psi\rangle)}.$$

But by assumption $\|(U_t^{(k)*} - U_t^*)\psi_0\| \to 0$ as $k \to \infty$ for all $\psi_0 \in \mathcal{H}_0 \otimes \mathcal{F}$, so that in particular $\langle \psi, U_t^{(k)*} U_t \psi \rangle \to \|\psi\|^2$ as $k \to \infty$. We thus obtain

$$\|(T_t^{(k;\alpha\beta)} - T_t^{(\alpha\beta)})v\| \xrightarrow{k\to\infty} 0 \qquad \forall \alpha, \beta \in \mathbb{C}^n, \; v \in \mathcal{H}_0, \; t < \infty.$$

Now denote the resolvents $R_\lambda^{(k;\alpha\beta)} = (\lambda - \mathscr{L}^{(k;\alpha\beta)})^{-1}$. We fix $\lambda > 0$, $u \in \mathcal{D}_0$, and define

$$u^{(k)} = \int_0^\infty e^{-\lambda t} T_t^{(k;\alpha\beta)}(\lambda - \mathscr{L}^{(\alpha\beta)})u \, dt = R_\lambda^{(k;\alpha\beta)}(\lambda - \mathscr{L}^{(\alpha\beta)})u.$$

Then it follows as in the proof of (a)$\Rightarrow$(b) that $u^{(k)} \to u$ and $\mathscr{L}^{(k;\alpha\beta)}u^{(k)} \to \mathscr{L}^{(\alpha\beta)}u$ as $k \to \infty$ (in particular, $u^{(k)} \in \mathrm{Dom}(\mathscr{L}^{(k;\alpha\beta)})$). This establishes the claim.

It remains to show that if $\mathcal{D}$ is a core for all $\mathscr{L}^{(k;\alpha\beta)}$, $k \in \mathbb{N}$, then we may choose $\{u^{(k)}\} \subset \mathcal{D}$. Indeed, let us fix $\alpha, \beta \in \mathbb{C}^n$, $u \in \mathcal{D}_0$, and construct the sequence $u^{(k)}$ as before. As $\mathcal{D}$ is a core for $\mathscr{L}^{(k;\alpha\beta)}$, we find that $\{(v, \mathscr{L}^{(k;\alpha\beta)}v) : v \in \mathcal{D}\}$ is dense in $\mathrm{Graph}(\mathscr{L}^{(k;\alpha\beta)})$. Therefore, for every k, we can find a vector $w^{(k)} \in \mathcal{D}$ such that

$$\|w^{(k)} - u^{(k)}\| < k^{-1}, \qquad \|\mathscr{L}^{(k;\alpha\beta)}w^{(k)} - \mathscr{L}^{(k;\alpha\beta)}u^{(k)}\| < k^{-1}.$$

Then $w^{(k)} \to u$ and $\mathscr{L}^{(k;\alpha\beta)}w^{(k)} \to \mathscr{L}^{(\alpha\beta)}u$, and the proof is complete. $\qquad\square$

5. Proof of the singular perturbation theorem

Before devoting ourselves to the proof of theorem 3, let us show that the coefficients in assumption 3 satisfy the unitarity relations in condition 2.

Proof of Lemma 2. We must verify the following relations:

$$K + K^\dagger = -\sum_{i=1}^n L_i L_i^\dagger, \qquad M_i = -\sum_{j=1}^n N_{ij} L_j^\dagger, \qquad \sum_{j=1}^n N_{mj} N_{\ell j}^\dagger = \sum_{j=1}^n N_{jm}^\dagger N_{j\ell} = \delta_{m\ell}$$

on $\mathcal{D}_0$. The problem is that products such as $L_i L_i^\dagger$ will contain terms such as $G_i P_0 G_i^\dagger$, which can not be related directly to the unitarity conditions for $K^{(k)}$, etc. The additional conditions in assumption 3 are chosen precisely to alleviate this problem, as they allow us to cancel the P_0 terms sandwiched inside the products. For example, as $P_0(G_i - A\tilde{Y}F_i)P_1 = 0$,

$$\sum_{i=1}^n L_i L_i^\dagger = \sum_{i=1}^n P_0(G_i - A\tilde{Y}F_i)(G_i^\dagger - F_i^\dagger \tilde{Y}^\dagger A^\dagger)P_0.$$

Note that, by applying condition 1 and assumption 1 to $L_i^{(k)}$ and $K^{(k)}$,

$$k^2(Y + Y^\dagger) + k(A + A^\dagger) + B + B^\dagger = K^{(k)} + K^{(k)\dagger} =$$

$$-\sum_{i=1}^n L_i^{(k)} L_i^{(k)\dagger} = -\sum_{i=1}^n (k^2 F_i F_i^\dagger + k F_i G_i^\dagger + k G_i F_i^\dagger + G_i G_i^\dagger),$$

and as this must hold for all k we obtain

$$Y + Y^\dagger = -\sum_{i=1}^n F_i F_i^\dagger, \qquad A + A^\dagger = -\sum_{i=1}^n (F_i G_i^\dagger + G_i F_i^\dagger), \qquad B + B^\dagger = -\sum_{i=1}^n G_i G_i^\dagger.$$

Note, in particular, that as $F_i^\dagger P_0 = 0$ (assumption 2(d))

$$(A + A^\dagger)P_0 = -\sum_{i=1}^n F_i G_i^\dagger P_0, \qquad P_0(A + A^\dagger) = -\sum_{i=1}^n P_0 G_i F_i^\dagger.$$

Thus

$$\sum_{i=1}^n L_i L_i^\dagger = P_0 \left[(A + A^\dagger)\tilde{Y}^\dagger A^\dagger + A\tilde{Y}(A + A^\dagger) - B - B^\dagger - A\tilde{Y}(Y + Y^\dagger)\tilde{Y}^\dagger A^\dagger \right] P_0.$$

Using that $P_0 A P_0 = 0$ (assumption 2(e)), it follows that this expression coincides with $-K - K^\dagger$ as required. We now proceed to proving that the relation for M_i holds. Note that

$$\sum_{j=1}^n N_{ij} L_j^\dagger = \sum_{j,\ell=1}^n P_0 W_{i\ell}(\delta_{\ell j} + F_\ell^\dagger \tilde{Y} F_j)(G_j^\dagger - F_j^\dagger \tilde{Y}^\dagger A^\dagger)P_0,$$

where we have used assumption (3) as above. Using again assumption (2) and the above relations, we obtain through straightforward manipulation

$$\sum_{j=1}^n (\delta_{\ell j} + F_\ell^\dagger \tilde{Y} F_j)(G_j^\dagger - F_j^\dagger \tilde{Y}^\dagger A^\dagger)P_0 = (G_\ell^\dagger - F_\ell^\dagger \tilde{Y} A)P_0,$$

so that the claim follows directly. Next, we turn to the relations for N_{ij}. We must establish

$$\sum_{j,a,b=1}^n P_0 W_{ma}(\delta_{aj} + F_a^\dagger \tilde{Y} F_j)(\delta_{bj} + F_j^\dagger \tilde{Y}^\dagger F_b)W_{\ell b}^\dagger P_0 = \delta_{m\ell}$$

on $\mathcal{D}_0$, where we have used assumption (3). Using assumption (2) and the above relations as before, we obtain through straightforward manipulation

$$\sum_{j=1}^n (\delta_{aj} + F_a^\dagger \tilde{Y} F_j)(\delta_{bj} + F_j^\dagger \tilde{Y}^\dagger F_b) = \delta_{ba}.$$

Hence the unitarity relation for W_{ij} establishes the claim. It remains to show that

$$\sum_{j,a,b=1}^n P_0(\delta_{b\ell} + F_\ell^\dagger \tilde{Y}^\dagger F_b)W_{jb}^\dagger W_{ja}(\delta_{am} + F_a^\dagger \tilde{Y} F_m)P_0 = \delta_{m\ell}.$$

Using the unitarity relation for W_{ij} we find that we must show

$$\sum_{a=1}^n P_0(\delta_{a\ell} + F_\ell^\dagger \tilde{Y}^\dagger F_a)(\delta_{am} + F_a^\dagger \tilde{Y} F_m)P_0 = \delta_{m\ell}.$$

But this was already established above, and we are done. $\square$

We now turn to the proof of theorem 3. We will apply the general convergence result, theorem 2, which leaves the problem of finding the appropriate sequences of vectors $u^{(k)}$. To this end, we employ a simple but cunning trick due to Kurtz [19].

Proof of Theorem 3. By theorem 2, it suffices to show that for every $\alpha, \beta \in \mathbb{C}^n$ and $u \in \mathcal{D}_0$, there exists a sequence $\{u^{(k)}\} \subset \mathcal{D}$ such that

$$u^{(k)} \xrightarrow{k \to \infty} u, \qquad \mathscr{L}^{(k;\alpha\beta)}u^{(k)} \xrightarrow{k \to \infty} \mathscr{L}^{(\alpha\beta)}u.$$

Let $v \in \mathcal{D}$, and note that in the current setting (3) reads

$$\mathscr{L}^{(k;\alpha\beta)}v = k^2\,Yv + k\left(\sum_{i=1}^n F_i\beta_i - \sum_{i,j=1}^n \alpha_i^* W_{ij}F_j^\dagger + A\right)v$$

$$+ \left(\sum_{i,j=1}^n \alpha_i^* W_{ij}\beta_j + \sum_{i=1}^n G_i\beta_i - \sum_{i,j=1}^n \alpha_i^* W_{ij}G_j^\dagger + B - \frac{|\alpha|^2 + |\beta|^2}{2}\right)v$$

$$:= (k^2\,Y + k\,A^{(\alpha\beta)} + B^{(\alpha\beta)})v.$$

We will seek $u^{(k)}$ of the form $u^{(k)} = u + k^{-1}u_1 + k^{-2}u_2$ with $u_1, u_2 \in \mathcal{D}$. Note that

$$\mathscr{L}^{(k;\alpha\beta)}u^{(k)} = k^2\,Yu + k(Yu_1 + A^{(\alpha\beta)}u) + Yu_2 + A^{(\alpha\beta)}u_1 + B^{(\alpha\beta)}u + o(1).$$

Evidently the proof is complete if we can choose u_1, u_2 in such a way that $\mathscr{L}^{(k;\alpha\beta)}u^{(k)} = \mathscr{L}^{(\alpha\beta)}u + o(1)$ for all $\alpha, \beta \in \mathbb{C}^n$ and $u \in \mathcal{D}_0$. The structural requirements (assumption 2) are chosen precisely so that this is the case. First, note that $Yu = 0$ by assumption 2(b), as $u \in \mathcal{D}_0$. This takes care of the quadratic term. Next, the linear term ($\propto k$) must clearly vanish. Hence it must be the case that

$$\mathrm{span}\{Au, W_{ij}F_j^\dagger u, F_i u : u \in \mathcal{D}_0,\ i, j = 1, \ldots, n\} \subset \{Yv : v \in \mathcal{D}\}.$$

But note that $W_{ij}F_j^\dagger u = 0$ for all $u \in \mathcal{D}_0$ by assumption 2(d), while assumptions 2(d,e) imply that $F_i = P_1 F_i$ and $AP_0 = P_1 AP_0$. Thus $F_i v = Y\tilde{Y}F_i v$ and $Av = Y\tilde{Y}Av$ for all $v \in \mathcal{D}_0$ by assumption 2(c), which establishes the claim. In particular, we will choose $u_1 = -\tilde{Y}A^{(\alpha\beta)}u$, which cancels the linear term. It remains to choose $u_2 \in \mathcal{D}$ so that

$$\lim_{k\to\infty} \mathscr{L}^{(k;\alpha\beta)}u^{(k)} = Yu_2 + (B^{(\alpha\beta)} - A^{(\alpha\beta)}\tilde{Y}A^{(\alpha\beta)})u = \mathscr{L}^{(\alpha\beta)}u.$$

Note that even if we did not know in advance the form of K, L_i, M_i, N_{ij}, we must require that $\mathscr{L}^{(\alpha\beta)}u \in \mathcal{H}_0$ for every $u \in \mathcal{D}_0$ in order that the limit equation leaves $\mathcal{H}_0$ invariant. Hence, if the limit of the slow dynamics exists, we must be able to choose u_2 so that

$$P_1 Yu_2 = -P_1(B^{(\alpha\beta)} - A^{(\alpha\beta)}\tilde{Y}A^{(\alpha\beta)})u.$$

But $P_1 Y = Y\tilde{Y}Y = YP_1 = Y$ by assumptions 2(b,c), so that evidently we must have

$$\mathscr{L}^{(\alpha\beta)}v = P_0(B^{(\alpha\beta)} - A^{(\alpha\beta)}\tilde{Y}A^{(\alpha\beta)})v =$$

$$P_0 \left\{ \sum_{i,j=1}^{n} \alpha_i^* \left(\sum_{\ell=1}^{n} W_{i\ell}(F_\ell^\dagger \tilde{Y} F_j + \delta_{\ell j}) + \sum_{\ell=1}^{n} F_j \tilde{Y} W_{i\ell} F_\ell^\dagger \right) \beta_j \right.$$

$$- \sum_{i,j=1}^{n} \alpha_i^* \left(W_{ij}(G_j^\dagger - F_j^\dagger \tilde{Y}A) - A\tilde{Y}W_{ij}F_j^\dagger \right)$$

$$+ \sum_{i=1}^{n} \left(G_i - A\tilde{Y}F_i - F_i\tilde{Y}A \right) \beta_i + B - A\tilde{Y}A - \frac{|\alpha|^2 + |\beta|^2}{2}$$

$$\left. - \left[\sum_{i=1}^{n} F_i\beta_i \right] \tilde{Y} \left[\sum_{i=1}^{n} F_i\beta_i \right] - \left[\sum_{i,j=1}^{n} \alpha_i^* W_{ij} F_j^\dagger \right] \tilde{Y} \left[\sum_{i,j=1}^{n} \alpha_i^* W_{ij} F_j^\dagger \right] \right\} v$$

for all $v \in \mathcal{D}_0$, $\alpha, \beta \in \mathbb{C}^n$. It remains to verify two things: that this expression does indeed define, for every $\alpha, \beta \in \mathbb{C}^n$, the generator of the semigroup $T_t^{(\alpha\beta)}$, and that Y is invertible on the range of $P_1(B^{(\alpha\beta)} - A^{(\alpha\beta)}\tilde{Y}A^{(\alpha\beta)})P_0$. But the latter follows immediately from assumption 2(c), so it remains to deal with the former.

In order for the above expression to define the generator of $T_t^{(\alpha\beta)}$, at the very least the last two terms must vanish—after all, they are quadratic in β_i and α_i^*, respectively, which is inconsistent with (3). However, this is immediate from assumption 2(d), as it implies that $F_j^\dagger P_0 = P_0 F_i = 0$. Simplifying further, we find that for all $v \in \mathcal{D}_0$

$$\mathscr{L}^{(\alpha\beta)}v = P_0 \left\{ \sum_{i,j,\ell=1}^{n} \alpha_i^* W_{i\ell}(F_\ell^\dagger \tilde{Y} F_j + \delta_{\ell j})\beta_j - \sum_{i,j=1}^{n} \alpha_i^* W_{ij}(G_j^\dagger - F_j^\dagger \tilde{Y}A) \right.$$

$$\left. + \sum_{i=1}^{n} (G_i - A\tilde{Y}F_i)\beta_i + B - A\tilde{Y}A - \frac{|\alpha|^2 + |\beta|^2}{2} \right\} v.$$

Thus evidently the coefficients of assumption 3 emerge naturally from our approach (by lemma 1), and the proof is complete by lemma 2 and conditions 1–3.   □

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